MGS 2026 — Category Theory

Exercises

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1 Sets and Warm-Up

The exercises in this section use only sets and functions; no category theory yet. Notation: $f\colon X \to Y$ for a function; $\mathit{id}_X\colon X \to X$ for the identity; $g \circ f$ for composition (apply f first).

Throughout, $\mathbb{N}$ denotes the natural numbers $\{0, 1, 2, \ldots\}$ with successor $s(n) = n + 1$.

Exercise 1.1 (Isomorphisms) Let $f\colon X \to Y$ be a function.

(a) Show that if $g, h\colon Y \to X$ both satisfy $g \circ f = \mathit{id}_X$, $f \circ g = \mathit{id}_Y$ and $h \circ f = \mathit{id}_X$, $f \circ h = \mathit{id}_Y$, then $g = h$. Conclude: the two-sided inverse f^{-1} is unique when it exists.

(b) Show that f has a two-sided inverse if and only if f is a bijection.

(c) Prove: if $f\colon X \to Y$ and $g\colon Y \to Z$ are isomorphisms, so is $g \circ f$, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

(d) Prove: id_X is an isomorphism and $\mathit{id}_X^{-1} = \mathit{id}_X$.

Exercise 1.2 (Terminal sets) A set T is called *terminal* if for every set X there is exactly one function $X \to T$.

(a) Show that T is terminal if and only if T has exactly one element (i.e. T is a singleton in the usual sense).

(b) Show that any two terminal sets are isomorphic, without appealing to (a).

Exercise 1.3 (Initial sets) A set I is called *initial* if there is exactly one function $I \to X$ for every set X. Verify that a set is initial if and only if it is empty.

Exercise 1.4 (Kuratowski pairing and Cartesian product) The *Kuratowski definition* sets $(x, y) := \big\{\{x\}, \{x, y\}\big\}$ and $X \times Y := \{(x, y) \mid x \in X,\ y \in Y\}$ with $\mathit{fst}(x, y) = x$, $\mathit{snd}(x, y) = y$.

(a) Show that $(x, y) = (x', y')$ implies $x = x'$ and $y = y'$. (*Hint*: consider the cases $x = y$ and $x \neq y$ separately.)

(b) Prove the universal property: for every set Z and functions $f\colon Z \to X$, $g\colon Z \to Y$, there exists a unique $h\colon Z \to X \times Y$ with $\mathit{fst} \circ h = f$ and $\mathit{snd} \circ h = g$.

Exercise 1.5 (Monomorphisms and epimorphisms) A function $f\colon X \to Y$ is called a *monomorphism* (mono) if for all sets Z and functions $g, h\colon Z \to X$, $f \circ g = f \circ h$ implies $g = h$; and an *epimorphism* (epi) if for all sets Z and functions $g, h\colon Y \to Z$, $g \circ f = h \circ f$ implies $g = h$.

(a) Show f is a mono iff f is injective.

(b) Show f is an epi iff f is surjective.

Now let $(P, \leq)$ and $(Q, \leq)$ be posets and $f\colon P \to Q$ a monotone map. Call f a *mono in* **Pos** (resp. *epi in* **Pos**) if the same conditions hold with all sets and functions replaced by posets and monotone maps.

(c) Show f is a mono in **Pos** iff f is injective.

(d) Show f is an epi in **Pos** iff f is surjective. (*Hint*: for the non-trivial direction, suppose $q_0 \notin f(P)$ and take the discrete poset $\{0,1\}$ as codomain.)

An epi $f\colon P \to Q$ in **Pos** *splits* if there exists a monotone map $g\colon Q \to P$ with $f \circ g = id_Q$.

(e) ($\star$) Let $P = \{a,b,c\}$ with $a < b$ and c incomparable to both, and let $Q = \{0 < 1 < 2\}$. Define $f\colon P \to Q$ by $f(a) = 0$, $f(b) = 1$, $f(c) = 2$. Show f is surjective (hence epi by (d)), but does not split.

Exercise 1.6 (Structural recursion on $\mathbb{N}$) For any set A, element $a \in A$, and function $h\colon A \to A$, there is a unique $f\colon \mathbb{N} \to A$ with

$$f(0) = a, \qquad f(n{+}1) = h(f(n)).$$

(a) Use induction to show that such f is unique.

(b) Define $\mathsf{double}\colon \mathbb{N} \to \mathbb{N}$ by structural recursion and prove $\mathsf{double}(n) = 2n$ by induction.

(c) Show that *strong recursion* is derivable from structural recursion: for any set X, function $g\colon X \to A$, and function $h\colon A \times X \to A$, there is a unique $f\colon \mathbb{N} \times X \to A$ with

$$f(0,x) = g(x), \qquad f(n{+}1,x) = h(f(n,x),\, x).$$

(*Hint*: run recursion over A^X as A.)

(d) Using strong recursion, define addition of two numbers $\mathsf{add}\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

(e) Show that primitive recursion is derivable from strong recursion: for any set X, function $g\colon X \to A$, and function $h\colon A \times \mathbb{N} \times X \to A$, there is a unique $f\colon \mathbb{N} \times X \to A$ with

$$f(0,x) = g(x), \qquad f(n{+}1,x) = h(f(n,x),\, n,\, x).$$

(*Hint*: run strong recursion over $A \times \mathbb{N}$, instead of A.)

(f) Using primitive recursion, define $\mathsf{factorial}\colon \mathbb{N} \to \mathbb{N}$.

Recall that $\mathsf{List}(X)$ denotes the set of finite lists over a set X, with empty list $[\,]$ and cons $x :: xs$.

Exercise 1.7 (Structural recursion on $\mathsf{List}(X)$) For any set A, element $a \in A$, and function $h\colon X \times A \to A$, there is a unique $f\colon \mathsf{List}(X) \to A$ with

$$f([\,]) = a, \qquad f(x :: xs) = h(x,\, f(xs)).$$

(a) Define $\mathsf{length}\colon \mathsf{List}(X) \to \mathbb{N}$ by choosing appropriate a and h.

(b) Define concatenation $+\!\!+\colon \mathsf{List}(X) \times \mathsf{List}(X) \to \mathsf{List}(X)$ by recursion on the first argument.

(c) Prove by induction: $\mathsf{length}(xs +\!\!+ ys) = \mathsf{length}(xs) + \mathsf{length}(ys)$.

(d) ($\star$) State and prove the analogue of primitive recursion for $\mathsf{List}(X)$: a recursion scheme where the step function also has access to the tail xs, not just the head x and the recursive result $f(xs)$.

2 Categories and Functors

Exercise 2.1 (Poset as category) Let $(P, \leq)$ be a poset, viewed as a category (objects = elements, unique morphism $x \to y$ iff $x \leq y$).

(a) What are the isomorphisms in this category?

(b) What are the terminal and initial objects (if they exist)?

(c) What is the opposite category $(P, \leq)^{\mathsf{op}}$?

(d) A functor $(P, \leq) \to (Q, \leq)$ is exactly a … ? Prove it.

Exercise 2.2 (Isomorphisms) Let $\mathcal{C}$ be a category.

(a) Show that any morphism has *at most one* two-sided inverse.

(b) Show that the relation $A \cong B$ (there exists an isomorphism $A \to B$) is an equivalence relation on objects.

(c) Suppose $f\colon A \to B$ has a left inverse g (i.e. $g \circ f = id_A$) and a right inverse h (i.e. $f \circ h = id_B$). Show $g = h$ and hence f is an isomorphism.

Exercise 2.3 (Binary Products/Coproducts)

(a) Show that binary products are associative up to isomorphism: $(A \times B) \times C \cong A \times (B \times C)$ via a canonical isomorphism.

(b) Derive, by duality (apply (a) to $\mathcal{C}^{\mathsf{op}}$), that binary coproducts are associative: $(A+B)+C \cong A + (B + C)$.

Exercise 2.4 (Functors preserve isomorphisms) Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor and $f\colon A \to B$ an isomorphism in $\mathcal{C}$. Show that $F(f)\colon F(A) \to F(B)$ is an isomorphism in $\mathcal{D}$, with $F(f)^{-1} = F(f^{-1})$.

Exercise 2.5 (Full and faithful functors)

(a) Show that the forgetful functor $U\colon \mathbf{Mon} \to \mathbf{Set}$ is faithful but not full. (Hint for not-full: exhibit two monoids and a function between their underlying sets that is not a homomorphism.)

(b) Show that a fully faithful functor reflects isomorphisms: if $F(f)$ is an isomorphism, then so is f.

(c) ($\star$) Let $F\colon \mathcal{C} \to \mathcal{D}$ be full, faithful and surjective on objects. Show that F is an equivalence of categories: there exists $G\colon \mathcal{D} \to \mathcal{C}$ with $GF \cong \mathrm{Id}_{\mathcal{C}}$ and $FG \cong \mathrm{Id}_{\mathcal{D}}$.

Exercise 2.6 (List functor) Let $\mathsf{List}(X)$ denote the set of finite lists over X. For a function $f\colon X \to Y$, define $\mathsf{List}(f)\colon \mathsf{List}(X) \to \mathsf{List}(Y)$ by applying f pointwise: $\mathsf{List}(f)(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n))$.

(a) Show that $\mathsf{List}\colon \mathbf{Set} \to \mathbf{Set}$ is a functor.

(b) Show that $\mathsf{List}(X)$ carries a monoid structure under list concatenation (with the empty list as unit), and that $\mathsf{List}(f)$ is a monoid homomorphism. Conclude that $\mathsf{List}\colon \mathbf{Set} \to \mathbf{Mon}$ is a functor.

3 Natural Transformations and Exponentials

Exercise 3.1 (Exponential objects) Let $\mathcal{C}$ be a category with binary products. Recall that an *exponential* B^A is an object together with $\mathsf{ev}\colon B^A \times A \to B$ such that for every $f\colon C \times A \to B$ there is a unique $\mathsf{curry}(f)\colon C \to B^A$ with $\mathsf{ev} \circ (\mathsf{curry}(f) \times id_A) = f$.

(a) Verify that in **Set** the exponential B^A is the set of functions $A \to B$, with $\mathsf{ev}(f, a) = f(a)$.

(b) Show that if B^A exists then $\mathsf{Hom}(C, B^A) \cong \mathsf{Hom}(C \times A, B)$ naturally in C.

(c) Deduce that $\mathsf{Hom}(1, B^A) \cong \mathsf{Hom}(A, B)$, where 1 is the terminal object.

(d) Show that exponentials do not exist in **Mon**. Observe that there is exactly one morphism $1 \to M$ in **Mon** for any M, so by (c) a necessary condition for B^A to exist is $|\mathsf{Hom}_{\mathbf{Mon}}(A, B)| = 1$.

Exercise 3.2 (Recursion in **Mon**) Let us restate the structural recursion principle from Exercise 1.6 for monoids: For any monoid M, morphism $a\colon 1 \to M$, and monoid morphism $h\colon M \to M$, there is a unique monoid morphism $f\colon \mathbb{N} \to M$ with $f(0) = a$ and $f(n{+}1) = h(f(n))$.

(a) Verify this. What is the unique such f explicitly?

(b) ($\star$) Show that strong recursion *fails* in **Mon**: exhibit monoids X and M, a monoid morphism $g\colon X \to M$, and a monoid morphism $h\colon M \times X \to M$ such that no monoid morphism $f\colon \mathbb{N} \times X \to M$ satisfies $f(0, x) = g(x)$ and $f(n{+}1, x) = h(f(n, x), x)$. (Thus, existence of exponentials is indeed a critical step of the proof in Exercise 1.6(c))

(*Hint*: try $X = M = (2, +_2, 0)$, $g = id$, $h(m, x) = m +_2 x$, where $+_2$ is addition modulo 2.)

Exercise 3.3 (CCC is distributive) Let $\mathcal{C}$ be a cartesian closed category with coproducts. Show that $\mathcal{C}$ is *distributive*: for all objects A, B, C,

$$A \times (B + C) \;\cong\; (A \times B) + (A \times C).$$

(*Hint*: use the natural bijection $\mathsf{Hom}(A \times X, Y) \cong \mathsf{Hom}(X, Y^A)$ and the universal property of coproducts to construct mutually inverse morphisms.)

Exercise 3.4 (($\star$) Contravariant powerset)

(a) Show there is a bijection between subsets $A \subseteq X$ and characteristic functions $\chi_A\colon X \to 2$, $\chi_A(x) = 1$ iff $x \in A$.

(b) Under this bijection, show that $\mathsf{Hom}_{\mathbf{Set}}(f, 2)(g) = g \circ f$ corresponds to the inverse image $f^{-1}[-]$.

(c) Conclude that the contravariant powerset functor $\mathcal{Q}\colon \mathbf{Set}^{\mathsf{op}} \to \mathbf{Set}$ sending $f\colon X \to Y$ to $f^{-1}[-]\colon \mathcal{Q}(Y) \to \mathcal{Q}(X)$ is naturally isomorphic to $\mathsf{Hom}_{\mathbf{Set}}(-, 2)$.

Exercise 3.5 (Hom functors and Yoneda embedding) Let $\mathcal{C}$ be a category.

(a) Fix $A \in \mathcal{C}$. Show that $\mathsf{Hom}(A, -)\colon \mathcal{C} \to \mathbf{Set}$, sending $B \mapsto \mathsf{Hom}(A, B)$ and $f \mapsto f \circ -$, is a functor.

(b) Fix $B \in \mathcal{C}$. Show that $\mathsf{Hom}(-, B)\colon \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$, sending $A \mapsto \mathsf{Hom}(A, B)$ and $f \mapsto - \circ f$, is a functor.

(c) ($\star$) Show that the *Yoneda embedding* $\mathsf{y}\colon \mathcal{C} \to [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$, defined by $\mathsf{y}(A) = \mathsf{Hom}(-, A)$ and $\mathsf{y}(f) = \mathsf{Hom}(-, f)$, is a functor. (Here $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$ is the functor category; morphisms are natural transformations.)