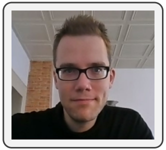
Towards a Higher-Order Bialgebraic **Denotational** Semantics

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Categorical Operational Semantics Project

- **Main idea:** given operational specification (set of rules), devise methods/properties of language, using category theory as leverage

  - **Methods:** abstract logical relations, abstract Howe's method, ...
  - **Properties:** compositionality, safety, adequacy, equivalence of programs, semantic styles
- **Side-effect:** categorical methods $\rightsquigarrow$ functional implementation (Haskell, Agda, Rocq)

Operational and Denotational Semantics

- Operational: how program p executes — transition system
- Denotational: what program p means — mathematical object $[\![p]\!]$

Two desiderata for $[\![-]\!]$:

1. Compositionality: $[\![\mathsf{f}(p_1,\ldots,p_n)]\!]$ depends only on f and $[\![p_1]\!],\ldots,[\![p_n]\!]$
2. Adequacy: $[\![p]\!] = [\![q]\!]$ implies p and q behaviourally equivalent

? Can denotational model be read off from operational rules?

? Can adequacy be proven henceforth?

Outline

Higher-order abstract GSOS

Locally final coalgebras

Construction in M-categories

Further examples

Conclusions

Higher-Order Abstract GSOS

First-Order Abstract GSOS

Turi and Plotkin's abstraction of GSOS rule format (Turi & Plotkin 1997):

- Signature endofunctor Σ
- Behaviour endofunctor B
- GSOS law — natural transformation $\rho_X \colon \Sigma(X \times BX) \to B(\Sigma^\star X)$

Example (process algebra):

- $\Sigma = \{\,|\colon 2, \varnothing\colon 0\} \cup \{a.(\text{-})\colon 1 \mid a \in A\}$
- $BX = \mathcal{P}_\omega(A \times X)$
- ρ encodes rules like:

$$\frac{p \xrightarrow{a} p'}{p \mid q \xrightarrow{a} p' \mid q}$$

First-Order Abstract GSOS

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- $\Sigma = \{ \mid : 2, \emptyset : 0\} \cup \{a.(-) : 1 \mid a \in A\}$
- $BX = \mathcal{P}_\omega(A \times X)$
- ρ encodes rules like:

behaviour from B

$$\frac{p \xrightarrow{a} p'}{p \mid q \xrightarrow{a} p' \mid q}$$

operation from Σ

Bialgebras: Operational and Denotational Model

- ρ-bialgebra = algebra $a\colon \Sigma A \to A$ + coalgebra $c\colon A \to BA$ + coherence (bialgebra law)
- Initial algebra $(\mu\Sigma, \iota)$ of program terms extends to initial bialgebra, with $\gamma\colon \mu\Sigma \to B\mu\Sigma$ — operational model, inducing bisimilarity $\sim$
- Final coalgebra νB extends to final bialgebra — denotational model

$$\begin{tikzcd}
\Sigma(\mu\Sigma) \arrow[r, "\cong"] \arrow[d, "\Sigma[\![-]\!]_\rho"'] & \mu\Sigma \arrow[r, "\gamma"] \arrow[d, dashed, "[\![-]\!]_\rho"] & B(\mu\Sigma) \arrow[d, "B[\![-]\!]_\rho"] \\
\Sigma(\nu B) \arrow[r, "\alpha"] & \nu B \arrow[r, "\cong"] & B(\nu B)
\end{tikzcd}$$

Compositionality holds by construction —
$[\![-]\!]_\rho$ is Σ-algebra morphism, hence $p \sim q$ implies $C[p] \sim C[q]$ for any context C

Adequacy trivial — denotational and behavioural equivalence coincide: $p \sim q$ iff $[\![p]\!]_\rho = [\![q]\!]_\rho$

Higher-Order GSOS

GSOS laws do not reach higher-order: program cannot consume other programs; this leads to:

Definition: Higher-order GSOS law in category $\mathcal{C}$ consists of

- Signature functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$
- Behaviour bifunctor $B\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$
- Family

$$\rho_{X,Y}\colon \Sigma\big(X \times B(X, Y)\big) \longrightarrow B\big(X, \Sigma^\star(X + Y)\big)$$

  natural in Y, dinatural in X

First argument of B is contravariant: programs can be evaluated on other programs

Typed Combinatory Logic xTCL

- Types Ty defined by grammar $\tau ::= \mathsf{u} \mid \tau \to \tau$
- Constructors
  - Unit type element: $\mathsf{e} \colon \mathsf{u}$
  - Application: $(\tau_1 \to \tau_2) \times \tau_1 \to \tau_2$
  - Combinators: $I_\tau \colon \tau \to \tau$, $K_{\tau_1,\tau_2} \colon \tau_1 \to (\tau_2 \to \tau_1)$, ...
  - "Multiary combinators" K'_{τ_1,τ_2}, $S'_{\tau_1,\tau_2,\tau_3}$, $S''_{\tau_1,\tau_2,\tau_3}$ for partial evaluation
- Rules:

$$\frac{}{I \xrightarrow{t} t} \qquad \frac{}{K \xrightarrow{t} K'(t)} \qquad \frac{}{K'(p) \xrightarrow{t} p} \qquad \frac{}{\mathsf{e} \xrightarrow{\checkmark}}$$

$$\frac{}{S \xrightarrow{t} S'(t)} \qquad \frac{}{S'(p) \xrightarrow{t} S''(p,t)} \qquad \frac{}{S''(p,q) \xrightarrow{t} (p\ t)(q\ t)} \qquad \frac{p \to p'}{p\ q \to p'\ q} \qquad \frac{p \xrightarrow{q} p'}{p\ q \to p'}$$

- Behaviour functor in $\mathbf{Set}^{\mathsf{Ty}}$: $B_{\mathsf{u}}(X,Y) = Y_{\mathsf{u}} + 1$, $B_{\tau_1\to\tau_2}(X,Y) = Y_{\tau_1\to\tau_2} + Y_{\tau_2}^{X_{\tau_1}}$

Rules as Higher-Order GSOS Law

Rules

$$\frac{s \to s'}{s\ t \to s'\ t} \qquad \frac{s \xrightarrow{t} s'}{s\ t \to s'}$$

correspond to

$$\rho((s, s')\ (t, \_)) = s'\ t$$
$$\rho((s, f)\ (t, \_)) = f(t)$$

$$\left(\rho_{X,Y} \colon \Sigma(X \times (Y + Y^X)) \to \Sigma^\star(X + Y) + (\Sigma^\star(X + Y))^X\right)$$

Higher-Order Bialgebras

- Higher-order ρ-bialgebra: $a\colon \Sigma A \to A$, + $c\colon A \to B(A, A)$ + higher-order bialgebra law
- Initial algebra $(\mu\Sigma, \iota)$ extends to initial bialgebra, with $\gamma\colon \mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$ — operational model

  ! Final bialgebra does not exist, already for xTCL (shown in paper): $\sim$ is kernel of coit$(\gamma)\colon \mu\Sigma \to \nu\gamma.\, B(\mu\Sigma, \gamma)$, no longer of map into denotational model
- For xTCL, induced $\sim$ is strong applicative bisimilarity
- Compositionality of $\sim$ — $p \sim q$ implies $C[p] \sim C[q]$ — still can be shown in categorical generality

Fixpoints of Functors

- Initial algebra of programs $\mu\Sigma$ ($\Rightarrow$ initial bialgebra) is constructed inductively bottom-up:
  - $\emptyset$ — empty set
  - $\Sigma\emptyset$ — constants
  - $\ldots$
  - $\mu\Sigma = \bigcup_\kappa \Sigma^\kappa\emptyset$ — all terms
- Final coalgebra of behaviours ($\Rightarrow$ final bialgebra) for unary B — coinductively top-down:
  - Form 1, $B1$, $B^2 1$, $\ldots$ and then $\nu B = \lim_\kappa B^\kappa 1$
  - Example: for $BX = \mathcal{P}_\omega(A \times X)$ — finitely branching LTS, νB — synchronization trees, denotational domain for process algebra
- Mixed variance B cannot be simply iterated, because we are "growing" along one argument and "shrinking" along other
- Moreover, for mixed variance, final coalgebras need not yield final bialgebras

Adequacy for HO-GSOS

Our Solution: Locally Final Coalgebras

Definition: Coalgebra $z\colon Z \to B(Z,Z)$ is locally final if it is final coalgebra for endofunctor $B(Z,-)\colon \mathcal{C} \to \mathcal{C}$

- z is isomorphism $Z \cong B(Z,Z)$ (Lambek lemma)
- Need not be unique up to isomorphism of higher-order coalgebras — but often is

Example: xTCL, where $B_{\tau_1 \to \tau_2}(X,Y)$ mentions X only at τ_1, so types decrease along contravariant argument, and Z is built by recursion on types

- $Z_{\mathsf{u}} \cong Z_{\mathsf{u}} + 1 \cong \mathbb{N} + \{\infty\}$ — steps to $\checkmark$, or divergence
- $Z_{\tau_1 \to \tau_2} \cong Z_{\tau_1 \to \tau_2} + Z_{\tau_2}^{Z_{\tau_1}} \cong \mathbb{N} \times Z_{\tau_2}^{Z_{\tau_1}} + \{\infty\}$ — steps, then function $Z_{\tau_1} \to Z_{\tau_2}$, or divergence

Adequacy and Compositionality

Theorem: Every ρ-bialgebra is adequate: behaviour of program is determined by its denotation

ρ relatively flat — restriction bounding how deeply operations nest on right-hand sides of rules

- holds for xTCL and xCL (untyped version)
- typically holds for terminating languages
- fails for unbounded recursion: recursive types, or Y-combinator $\mathsf{Y}\,e \to e\,(\mathsf{Y}\,e)$

Theorem: For relatively flat ρ, every locally final coalgebra Z extends to ρ-bialgebra. By initiality, $[\![-]\!]_\rho \colon \mu\Sigma \to Z$ — compositional

Constructing Locally Final Coalgebras

Existence via M-Categories

For xTCL, Z was built by recursion on types. Metric enrichment abstracts this: distance measures size of smallest types on which morphisms differ

- M-category (after Birkedal et al.): hom-sets carry complete ultrametric
- F locally non-expansive if it does not increase distances
- F locally contractive if it properly shrinks distances

Theorem: If* B is locally contractive in the first argument, and locally non-expansive in the second, locally final coalgebra of B exists; it is unique when all hom-sets are inhabited

Proof Idea: Compute final coalgebra over second argument, then unique fixpoint in first

Example: For xTCL: $\mathcal{C} = \mathbf{Set}^{\mathsf{Ty}}$, $|\mathsf{u}| = 1$, $|\tau_1 \to \tau_2| = |\tau_1| + |\tau_2|$:

$$d(f,g) = \sup\{2^{-|\tau|+1} \mid f_\tau \neq g_\tau\}$$

agreement up to larger types $\Rightarrow$ smaller distance

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*essentially

Further Examples

In $\mathcal{C} = \mathbf{Set}^{\mathsf{Ty}}$:

- xPTCL — probabilistic xTCL: $B^{\mathsf{p}}_{\mathsf{u}}(X, Y) = \mathcal{D}_\omega(Y_{\mathsf{u}}) + 1$, ...

Untyped languages have no types to induce contractivity — work in topos of trees $\mathbf{Set}^{\mathbb{F}}$, where later modality $\blacktriangleright$ makes B contractive:

- xCL — untyped combinatory logic: locally final coalgebra consists of guarded interaction trees, known class of denotational models, emerging here from operational semantics
- xNCCL — non-deterministic concurrent xCL: branching interaction trees
- λ-calculus in topos of variable trees $\mathcal{S}_{\mathbb{F}} \cong \mathbf{Set}^{\mathbb{F}\times\omega^{\mathsf{op}}}$, with guarded substitution component
$$B^{\lambda}(X, Y) = \langle\!\langle \blacktriangleright X, \blacktriangleright Y \rangle\!\rangle \times (\blacktriangleright Y + \blacktriangleright Y^{\blacktriangleright X} + 1)$$

Conclusions and Future Work

Denotational semantics for higher-order languages need not be conceptually isolated from operational semantics – it may follow from it, like Turi and Plotkin intended

Conclusions:

- Locally final coalgebras as denotational domains
- Compositionality and adequacy, generically, for every ρ-bialgebra
- Existence, uniqueness and construction via M-categories

Future work:

- Soundness and full abstraction
- Big-step semantics, weak applicative bisimilarity
- Order-theoretic domains: CPO-enriched categories
- π-calculi, substructural λ-calculi, relation to guarded recursive powerdomains of Møgelberg and Vezzosi

Thank you

ρ-Bialgebras

Bialgebra law for $\Sigma X \xrightarrow{a} X \xrightarrow{c} B(X,X)$ requires commutativity of

$$\begin{tikzcd}
\Sigma X \arrow[r, "a"] \arrow[d, "\Sigma\langle \mathsf{id}, c\rangle"'] & X \arrow[r, "c"] & B(X,X) \\
\Sigma(X \times B(X,X)) \arrow[rr, "\rho_{X,X}"] & & B(X, \Sigma^\star(X+X)) \arrow[u, "B(\mathsf{id}, \hat{a}\cdot\Sigma^\star\nabla)"']
\end{tikzcd}$$

Morphisms are simultaneously Σ-algebra and higher-order B-coalgebra morphisms

- Operational model $(\mu\Sigma, \iota, \gamma)$ is initial ρ-bialgebra
- Hence every ρ-bialgebra (Z, a, z) receives unique $[\![-]\!]_\rho \colon \mu\Sigma \to Z$, and $[\![-]\!]_\rho$ is Σ-algebra morphism: $[\![-]\!]_\rho$ is compositional

Strong Applicative Bisimilarity

For xTCL, strong applicative bisimilarity $\sim$ is greatest $R \subseteq \mu\Sigma \times \mu\Sigma$ closed under

$$\begin{tikzcd} p \arrow[r, dash, "R"] \arrow[d] & q \arrow[d, dashed] \\ p' \arrow[r, dash, dashed, "R"] & q' \end{tikzcd} \qquad \begin{tikzcd} p \arrow[r, dash, "R"] \arrow[d, "t"'] & q \arrow[d, dashed, "t"] \\ p' \arrow[r, dash, dashed, "R"] & q' \end{tikzcd} \qquad \begin{tikzcd} p \arrow[r, dash, "R"] \arrow[d, "\checkmark"'] & q \arrow[d, dashed, "\checkmark"] \\ {} & {} \end{tikzcd} \qquad + \text{ 3 dual ones}$$

Relative Flatness

Let $(J, <)$ be well-founded and $\Sigma = \coprod_{j \in J} \Sigma_j$

Relatively flat higher-order GSOS law is family

$$\rho^j_{X,Y} \colon \Sigma_j(X \times B(X,Y)) \to B(X,\ \Sigma^\star_{<j}(X+Y) + \Sigma_j \Sigma^\star_{<j}(X+Y))$$

Right-hand sides use only one layer of Σ_j on top of strictly smaller operations

xTCL: $J = \{0 < 1\}$, with $\Sigma_0 =$ application and Σ_1 everything else

Introduced for logical predicates (Goncharov et al. 2024) – exactly what makes denotational models work

Unbounded recursion breaks it: recursive types $\mu X.\tau$, or Y-combinator $\mathsf{Y}\, e \to e\, (\mathsf{Y}\, e)$

Extension Theorem: Bialgebra Morphism

By initiality of operational model, (Z, a_ρ, z) receives unique bialgebra morphism:

$$
\begin{tikzcd}
\Sigma(\mu\Sigma) \arrow[r, "\cong"] \arrow[dd, "\Sigma[\![-]\!]_\rho"'] & \mu\Sigma \arrow[r, "\gamma"] \arrow[dd, dashed, "[\![-]\!]_\rho"] & B(\mu\Sigma, \mu\Sigma) \arrow[dr, "B(\mathsf{id}, [\![-]\!]_\rho)"] & \\
 & & & B(\mu\Sigma, Z) \\
\Sigma Z \arrow[r, "a_\rho"] & Z \arrow[r, "\cong"] & B(Z, Z) \arrow[ur, "B([\![-]\!]_\rho, \mathsf{id})"'] &
\end{tikzcd}
$$

Left square is compositionality of $[\![-]\!]_\rho$

Adequacy: Factorization

Σ-algebra (A, a) is adequate if beh factorizes through $[\![-]\!]$:

$$\begin{tikzcd}
 & \mu\Sigma \arrow[dl, "{[\![-]\!]}"'] \arrow[dr, "\mathsf{beh}"] & \\
A \arrow[rr, dashed, "\exists"] & & \nu B(\mu\Sigma, -)
\end{tikzcd}$$

For xTCL, behavioural equivalence w.r.t. $\nu B(\mu\Sigma, -)$ is strong applicative bisimilarity

M-Categories

M-category is category whose hom-sets carry complete, 1-bounded ultrametrics, with non-expansive composition

- F is locally non-expansive / locally contractive if each $F_{X,Y}\colon \mathcal{C}(X,Y) \to \mathcal{C}(FX,FY)$ is; contractivity factor global
- Increasing Cauchy tower: $f_n\colon A_n \rightleftarrows A_{n+1}\colon g_n$ with $g_n \cdot f_n = \mathrm{id}$ and $d(f_n \cdot g_n, \mathrm{id}) \to 0$
- Stable: $d(f,g) = \sup_i d(h_i \cdot f, h_i \cdot g)$ for jointly monic (h_i)

$\mathbf{Set}^{\mathsf{Ty}}$ is stable M-category, for $|\mathsf{u}| = 1$ and $|\tau_1 \to \tau_2| = |\tau_1| + |\tau_2|$:

$$d(f,g) = \sup\{2^{-(|\tau|-1)} \mid \tau \in \mathsf{Ty} \text{ and } f_\tau \neq g_\tau\}$$

Fixed point theorem (after Birkedal et al.): locally contractive $F\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{C}$ on M-category with terminal object and inverse limits of Cauchy towers has fixed point

Existence of Locally Final Coalgebras

Theorem: Let $\mathcal{C}$ be stable M-category with terminal object and limits of chains, and $B\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$ such that

1. B is locally contractive in first and locally non-expansive in second argument;
2. $\mathcal{C}(1, B(1,1))$ is inhabited;
3. for each X, final sequence of $B(X, -)$ converges

Then B has locally final coalgebra; if all hom-sets are inhabited, it is unique up to isomorphism

Proof: $F\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{C}$, $FX = \nu B(X, -)$; fixed points of F are exactly locally final coalgebras of B; apply fixed point theorem

Construction: $Z = \lim_n F^n 1$. For xTCL, $(F^n 1)_\tau = Z_\tau$ for $n \geq |\tau|$ – ad hoc tree description