Relators and Notions of Simulation Revisited

Sergey Goncharov, Dirk Hofmann, Pedro Nora,

Lutz Schröder, Paul Wild

LICS 2025, June 23–26, 2025

Outline

Bisimulations for LTS

Bisimulations for Coalgebras

Difunctional Functoriality

coBarr Relators

Conclusions

Bisimulation for LTS

Recall: labelled transition system (LTS) $(X, (\xrightarrow{a})_{a \in \mathcal{A}})$ is

- state space X
- transition relations $\xrightarrow{a} \subseteq X \times X$ for every $a \in \mathcal{A}$

(Strong) bismilarity $\sim$ is greatest such relation r that

$$
\begin{tikzcd}
x \arrow[r, dash, "r"] \arrow[d, "a"'] & y \arrow[d, dashed, "a"] \\
x' \arrow[r, dash, dashed, "r"] & y'
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, dash, "r"] \arrow[d, dashed, "a"'] & y \arrow[d, "a"] \\
x' \arrow[r, dash, dashed, "r"] & y'
\end{tikzcd}
\qquad (a \in \mathcal{A})
$$

- It follows that $\sim$ is equivalence
- We understand it as behavioural equivalence
- Not necessarily greatest $r \rightsquigarrow$ bisimulation
- Only left square $\rightsquigarrow$ similarity/simulation

Is it the only "coinductive" way to define $\sim$? No!

Twisted Bisimulation

Assume $\mathcal{A} = \{a, b\}$. Twisted bisimulation r additionally includes alternative:

$$
\begin{tikzcd}
x \arrow[r, dash, "r"] \arrow[d, "a"'] & y \arrow[d, dashed, "b"] \\
x' \arrow[r, dash, dashed, "r"] & y'
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, dash, "r"] \arrow[d, "b"'] & y \arrow[d, dashed, "b"] \\
x' \arrow[r, dash, dashed, "r"] & y'
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, dash, "r"] \arrow[d, "b"'] & y \arrow[d, dashed, "a"] \\
x' \arrow[r, dash, dashed, "r"] & y'
\end{tikzcd}
$$

plus converses, plus duals

Again, $\sim$ is greatest such r

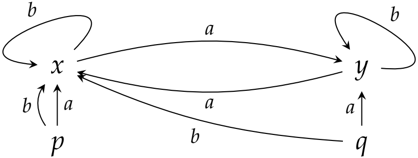
In more detail: notion of bisimulation is

- Sound if bisimilarity $\subseteq \sim$
- Complete if $\sim \subseteq$ bisimilarity

$\Longrightarrow$ twisted bisimulation is sound and complete

Twisted Bisimulation: Example

Consider LTS



- Smallest bisimulation in the usual sense relating p and q is

$$\{(p,q),(x,x),(x,y),(y,y),(y,x)\}$$

- Using twisted bisimulation, we can make do with the strictly smaller relation

$$\{(p,q),(x,x),(x,y),(y,y)\}$$

- So, twisted bisimulation is more permissive

Questions:

1. Is there most permissive one?
2. What is bisimulation anyhow?

Behavioural Equivalence Coalgebraically

- F-coalgebra (X, α) for an endofunctor $F\colon \mathbf{Set} \to \mathbf{Set}$ consists of a set X of states and a transition map $\alpha\colon X \to FX$
- $x \in X$ and $y \in Y$ of coalgebras (X, α) and (Y, β), respectively, are behaviourally equivalent if there exist a coalgebra (Z, γ), $f\colon X \to Z$, $g\colon Y \to Z$, such that

$$
\begin{tikzcd}
X \arrow[d, "\alpha"'] & Z \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\gamma"] & Y \arrow[d, "\beta"] \\
FX & FZ \arrow[l, "Ff"'] \arrow[r, "Fg"] & FY
\end{tikzcd}
$$

such that $f(x) = g(y)$

Example: $F = \mathcal{P}(A \times -)$ – LTS functor

Many more examples: Kripke frames, Markov chains, neighbourhood structures, weighted transitions systems, etc.

(Bi)simulation Coalgebraically

- Relator for functor $F \colon \mathbf{Set} \to \mathbf{Set}$ is monotone function on relations, such that

$$R(r \colon X \nrightarrow Y) \colon FX \nrightarrow FY$$

- Given relator R, relation $r \colon X \nrightarrow Y$ is R-simulation from coalgebra $\alpha \colon X \to FX$ to $\beta \colon Y \to FY$ if

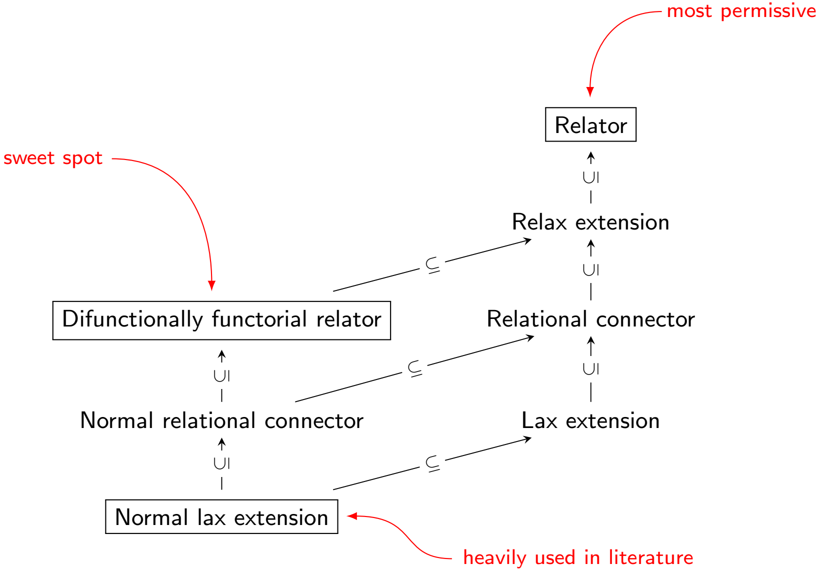
relation converse

$$r \leq \beta^{\circ} \cdot Rr \cdot \alpha, \qquad (*)$$

i.e, if $x\ r\ y$ entails $\alpha(x)\ Rr\ \beta(y)$, for all $x \in X$ and $y \in Y$

- R-bisimulation are R-simulation for symmetric relators, i.e. $R(r^{\circ}) = (Rr)^{\circ}$
- R-bisimilarity is defined as a greatest fixpoint of $(*)$ by Knaster-Tarski theorem

Relators and Lax-Extensions



Some Common Knowledge

- Barr relator $\bar{F}$ of a functor F takes a relation $r\colon X \nrightarrow Y$ as a span

$$X \xleftarrow{\pi_1} Z \xrightarrow{\pi_2} Y$$

  and returns

$$FX \xleftarrow{F\pi_1} FZ \xrightarrow{F\pi_2} FY$$

- $\bar{F}$-bisimulation is called Aczel-Mendler (bi)simulation
- $\bar{F}$-bisimulation is always sound, but need not be complete
- If F preserves weak pullbacks $\bar{F}$ is complete and is a normal (=identity-preserving) lax extension*
- Normal lax extension work more generally (e.g. for monotone neighbourhood functor)

But do we actually need lax extensions?

*Definition omitted

Difunctional Functoriality

F-relator R is difunctionally functorial if for all functions $f \colon X \to Z$ and $g \colon Y \to Z$, $R(g^\circ \cdot f) = (Fg)^\circ \cdot Ff$.

Result # 1: if R is difunctionally functorial then R-similarity is sound and complete

Difunctional functoriality is much weaker requirement than normal laxness

Result # 2: R is a lax extension iff the induced class of (bi)simulations contains all coalgebra homomorphisms and their converses and is closed under composition

Result # 3: Normality is essentially a necessary condition for soundness of bisimulations

coBarr Relator

Difunctional relations are those r that are presented by co-spans

$$X \xrightarrow{\iota_1} A \xleftarrow{\iota_2} Y$$

We then define coBarr relators via co-spans

$$FX \xrightarrow{F\iota_1} FA \xleftarrow{F\iota_2} FY$$

i.e. $\underline{F}r = (F\iota_2)^\circ \cdot F\iota_1$

However (!), this is only well-defined if $\underline{F}$ is independent of the choice of the cospan, which is iff F preserves $1/4$-iso pullbacks

$$\begin{tikzcd}
P \arrow[r] \arrow[d, tail, "\cong"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

Barr Relator v.s. coBarr Relator

- Recently*: preservation of $1/4$-iso pullbacks is necessary for admitting a normal lax extension
- If F weakly preserves pullbacks then $\bar{F}$ is least normal lax extension

Contrastingly:

Result # 4: If F preserves $1/4$-iso pullbacks then

1. $\underline{F}$ is symmetric difunctionally functorial relation $\implies$ $\underline{F}$-bisimulation is sound and complete ($1/4$-iso pullback preservation sufficient)
2. If F weakly preserves pullbacks, $\underline{F}r = \bar{F}\hat{r}$ where $\hat{r}$ – difunctional closure of r
3. $\underline{F}$ is greatest difunctionally functorial relator

*Goncharov, Hofmann, Nora, Schröder, Wild, "Identity-Preserving Lax Extensions and Where to Find Them", 2025.

Greatest Normal Lax Extension

Twisted bisimulation is induced not only by a difunctionaly functorial relator, but by greatest normal lax extension

Can we construct them in general?

Idea: lax extensions form a complete lattice, so we can form join of all normal lax extensions

Problems 1: The lattice may be empty
Problems 2: The joins must preserve normality

Result # 5: Largest normal lax extension exists whenever F preserves inverse images

$$\text{weak pb. preservation} \implies \text{inverse image preservation} \implies 1/4\text{-pb. preservation}$$

Conclusions

- We understood (normal) lax extensions through their compositionality properties
- Difunctionally functorial relators – novel modest condition ensuring soundness and completeness of coalgebraic simulation
- $1/4$-iso pullback preservation as fundamental property for constructing sound and complete relators
- Existence of largest normal lax extension for inverse image preserving functors

References