Probabilistic Strategies: Definability and the Tensor Completeness Problem

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Outline

Simple interaction model

Nondeterministic environments

Sums and Tensors

Probabilistic environments

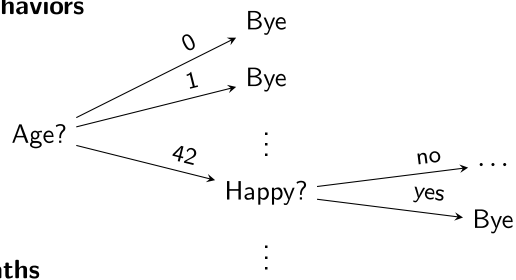
Results

Open problem

Simple Algebraic Language

$$M, N ::= \overset{\omega\text{-ary}}{\text{Age?}}(M_i)_{i\in\mathbb{N}} \mid \overset{\text{binary}}{\text{Happy?}}(M, N) \mid \overset{\text{0-ary (=constant)}}{\text{Bye}}$$

Terms = **Behaviors**



Traces = **Paths**

```
$ Age?
> 42
$ Happy?
> yes
$ Bye
```

More Generally

- Signature: family of sets $(\mathsf{Ar}(k))_{k\in\Sigma}$ where $\mathsf{Ar}(k) \in \mathbb{N} \cup \{\mathbb{N}\}$
- Terms, also Programs

$$M, N ::= \mathsf{Req}\, k?(M_i)_{i\in\mathsf{Ar}(k)}$$

- Traces, also Plays, are Q/A interaction sequences

$$\underbrace{k_1?}\ \underbrace{i_1{}^{\leq \mathsf{Ar}(k_1)}}\ \underbrace{k_2?}\ \underbrace{i_2{}^{\leq \mathsf{Ar}(k_2)}}\ \ldots\ \underbrace{k_n?} \longleftarrow \text{passive-ending}$$

and

$$\underbrace{k_1?}\ \underbrace{i_1{}^{\leq \mathsf{Ar}(k_1)}}\ \underbrace{k_2?}\ \underbrace{i_2{}^{\leq \mathsf{Ar}(k_2)}}\ \ldots\ \underbrace{k_n?}\ \underbrace{i_n{}^{\leq \mathsf{Ar}(k_n)}} \longleftarrow \text{active-ending}$$

- Programs generate traces; two programs are trace-equivalent if they generate same traces

Remarks:

1. Programs well-founded $\implies$ traces finite
2. Infinite arities $\implies$ no Kőnig's lemma $\implies$ well-founded $\neq$ finite
3. So far, trace equivalence = syntactic equality

Warm-Up: Non-Determimism

Non-Deterministic Programs

- Non-deterministic programs:

  $$M, N ::= \mathsf{Req}\, k?(M_i)_{i \in \mathsf{Ar}(k)} \mid M + N$$

  For countable non-determinism:

  $$M, N ::= \mathsf{Req}\, k?(M_i)_{i \in \mathsf{Ar}(k)} \mid \sum_{n \in \mathbb{N}} M_n$$

- Non-deterministic traces $\mathcal{Traces}\,(M)$ are defined analogously: sets of plays (=traces), program M can exhibit
- Trace equivalence: $\mathcal{Traces}\,(M) = \mathcal{Traces}\,(N)$
- More generally: non-deterministic strategy is a set of plays
  $+$ a coherence condition ($\approx$ prefix-closure)

Key questions:

1. Completeness: how to logically characterize trace equivalence?
2. Definability: how to characterize definable strategies $\mathcal{Traces}\,(M)$ among all strategies?

Non-Deterministic Programs

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$$M, N ::= \mathsf{Req}\, k?(M_i)_{i \in \mathsf{Ar}(k)} \mid \sum_{n \in \mathbb{N}} M_n$$

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Key questions:

1. $\boxed{\text{Completeness}}$ **!**: how to logically characterize trace equivalence?
2. Definability: how to characterize definable strategies $\mathcal{T}races\,(M)$ among all strategies?

Algebraic Theories: Sums and Tensors

Theory of finitary non-determinism

$(M+N)+K \equiv M+(N+K) \qquad M+N \equiv N+M \qquad M+M \equiv M$

- Sum of theories: join operations and equations
  $\Longrightarrow$ bisimulation equivalence, e.g.

$$\mathsf{Req}\,k?(\mathsf{Req}\,l?N + \mathsf{Req}\,r?M) \not\equiv \mathsf{Req}\,k?(\mathsf{Req}\,l?N) + \mathsf{Req}\,k?(\mathsf{Req}\,r?M)$$

- Tensor of theories[1]: additionally quotient by tensor laws:

$$\mathsf{Req}\,k?(M_i)_{i\in\mathsf{Ar}(k)} + \mathsf{Req}\,k?(N_i)_{i\in\mathsf{Ar}(k)} \equiv \mathsf{Req}\,k?(M_i+N_i)_{i\in\mathsf{Ar}(k)}$$

$\Longrightarrow$ trace equivalence (normalization by tensor law)

free monad on Σ

Remarks: In terms of monads: starting with signature Σ,

1. Monad coproduct $\Sigma^\star + \mathcal{P}_{<\omega}$ – for bisimulation equivalence
2. Monad tensor $\Sigma^\star \otimes \mathcal{P}_{<\omega}$ – for trace equivalence

[1] Freyd, "Algebra valued functors in general and tensor products in particular", 1966.

Convention

From now on, assume one binary operation

$$`*'$$

(so plays can be written as $?i_1?i_2\dots \quad (i_k \in \{0,1\}))$

Countable Non-determinism

Tensor Law for Countable Non-determinism (for $*$)

$$\sum_{n\in\mathbb{N}} M_n * N_n = \left(\sum_{n\in\mathbb{N}} M_n\right) * \left(\sum_{n\in\mathbb{N}} N_n\right)$$

Theorem: Tensor equivalence is (sound and) complete for trace equivalence[2].

Note: We cannot just normalize by tensor law, e.g.

$$x * x + x * (x * x) + x * (x * (x * x)) + \dots$$

would yield undefinable strategy with infinite play: ?1?1?1...

Proof Idea.

- Let $M \leq N$ if $M + N \equiv N$. Then $M \equiv N$ iff $M \leq N$ and $N \leq M$
- Prove that $\mathcal{T}races\,(M) \leq \mathcal{T}races\,(N)$ entails $M \leq N$ by induction on M $\square$

[2] Bowler, Levy, and Plotkin, "Initial Algebras and Final Coalgebras Consisting of Nondeterministic Finite Trace Strategies", 2018.

Educated Guess

Tensor equivalence must be sound and complete for trace equivalence

But is it?

This work: countably probabilistic programs –

- Yes, for finitary signatures
- For infinitary signatures – open problem

Probabilistic Traces

Countable Distributions

Tensor Law for Countable Probability (for $*$)

$$(\sum_{n\in\mathbb{N}} p_n \cdot M_n) * (\sum_{n\in\mathbb{N}} p_n \cdot N_n) \equiv \sum_{n\in\mathbb{N}} p_n \cdot (M_n * N_n)$$

where $\sum_{n\in\mathbb{N}} p_n = 1$, $\forall n \in \mathbb{N}.\, p_n \geq 0$

Laws for countable distributions = laws of super-convex algebras, extending familiar convex algebras

Probabilistic strategies: such functions σ from passive-ending plays to $[0,1]$ that

$$\sigma(\epsilon) = 1 \qquad\qquad \sigma(s) = \sum_{k\in\Sigma} \sigma(s\ i\ k?)$$

(generalization of prefix-closure)

Probabilistic Trace Semantics

- Each $k \in \Sigma$ yields semantic counterpart of Req $k?$ on strategies:

$$\mathcal{R}eq\, k?(\sigma_i)_{i \in \mathsf{Ar}(k)} \quad : \quad \begin{cases} k? & \mapsto \quad 1 \\ k?\ i\ s & \mapsto \quad \sigma_i(s) \\ l?\ i\ s & \mapsto \quad 0 \qquad \text{otherwise} \end{cases}$$

- Probabilistic choice extends pointwise
- We then define probabilistic trace semantics by structural recursion:

$$\begin{aligned} \mathcal{T}races\left(\mathsf{Req}\, k?(M_i)_{i \in \mathsf{Ar}(k)}\right) &= \mathcal{R}eq\, k?(\mathcal{T}races\,(M_i))_{i \in I} \\ \mathcal{T}races\,(M +_p N) &= \mathcal{T}races\,(M) +_p \mathcal{T}races\,(N) \\ \mathcal{T}races\left(\textstyle\sum_{n \in \mathbb{N}} p_n \cdot M_n\right) &= \textstyle\sum_{n \in \mathbb{N}} p_n \cdot \mathcal{T}races\,(M_n) \end{aligned}$$

Proposition: for finitary distributions, trace semantics is sound a complete

Proof: normalization by tensor law □

Example

Let

$$M = \tfrac{1}{2} x * z + \tfrac{1}{4} y * (x * z) + \tfrac{1}{8} y * (y * (x * z)) + \tfrac{1}{16} y * (y * (y * (x * z))) + \cdots$$
$$N = \tfrac{1}{2} y * z + \tfrac{1}{4} x * (y * z) + \tfrac{1}{8} x * (x * (y * z)) + \tfrac{1}{16} x * (x * (x * (y * z))) + \cdots$$

Then $\mathcal{Traces}\,(M) = \mathcal{Traces}\,(N)$,

Example

Let

$$M = \tfrac{1}{2} x * z + \tfrac{1}{4} y * (x * z) + \tfrac{1}{8} y * (y * (x * z)) + \tfrac{1}{16} y * (y * (y * (x * z))) + \cdots$$
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Then $\mathcal{T}races\,(M) = \mathcal{T}races\,(N)$,and in fact $M \equiv N$:

$$M \equiv \tfrac{1}{4} x * z + \tfrac{2}{8}(\tfrac{1}{2} x * z + \tfrac{1}{2} y * (x * z)) + \tfrac{3}{16}(\tfrac{1}{3} x * z + \tfrac{1}{3} y * (x * z) + \tfrac{1}{3} y * (y * (x * z)))$$

$\{$*completeness for finite distributions*$\}$

$$\equiv \tfrac{1}{4} x * z + \tfrac{2}{8}(\tfrac{1}{2} y * z + \tfrac{1}{2} x * (x * z)) + \tfrac{3}{16}(\tfrac{1}{3} y * z + \tfrac{1}{3} y * (y * z) + \tfrac{1}{3} x * (x * (x * z)))$$
$$\equiv \tfrac{1}{4} x * z + \tfrac{1}{8} x * (x * z) + \tfrac{1}{16} x * (x * (x * z))) + \cdots +$$
$$\tfrac{1}{4} y * z + \tfrac{1}{8} y * (y * z) + \tfrac{1}{16} y * (y * (y * z))) + \cdots$$

and symmetrically for N

Idea of Solution

- Key step: propagating choices upwards with

$$\left(\sum_{n\in I} p_n \cdot M_n\right) * \left(\sum_{m\in J} N_m \cdot t_m\right) \rightsquigarrow \sum_{n\in I, m\in J} (p_n \cdot q_m) \cdot N_n * M_m$$

- W.l.o.g. we then can start with

$$\mathcal{T}races\,(M) = \mathcal{T}races\,(N)$$

  where $M = \sum_{n\in\mathbb{N}} p_n \cdot M_n$ with choice-free M_n and same for N
- Find first such k that $\sum_{n\leq k} p_n$ accedes $1/2$
- Find first such m that $\sum_{n\leq m} q_n \cdot N_n$ "accedes" $\sum_{n\leq k} p_n \cdot M_n$
- Prove that $\mathcal{T}races\left(\sum_{n\leq m} q_n \cdot M_n\right) \setminus \mathcal{T}races\left(\sum_{n\leq k} p_n \cdot N_n\right)$ is definable, yielding $\Delta = \Delta_0$, such that

$$\sum_{n\leq m} q_n \cdot M_n \equiv \sum_{n\leq k} p_n \cdot N_n + \Delta_0$$

- Propagate Δ infinitely, alternating between N and M

Glimpse at Definability

A strategy σ is definable if $\sigma = \mathcal{T}races\,(M)$ for some M

Proposition: In language with $*/2$ and arbitrary constants X, σ is definable iff for every $(b_1 b_2 \ldots) \in 2^\omega$,

$$\sum\{\sigma(?b_1 \ldots ?b_n\ x) \mid n \in \mathbb{N}, x \in X\} = 1$$

Intuitively, σ is definable if there are no emergent infinite traces

Example:

$$\begin{aligned}&\{x \mapsto \tfrac{1}{2}, ?0x \mapsto \tfrac{1}{2}, ?1x \mapsto \tfrac{1}{4}, ?1?0x \mapsto \tfrac{1}{8}, ?1?1x \mapsto \tfrac{1}{8}, \ldots\}\\ &= \ \mathcal{T}races\left(\tfrac{1}{2}x + \tfrac{1}{4}x * x + \tfrac{1}{8}x * (x * x) + \ldots\right)\end{aligned}$$

but not

$$\{x \mapsto \tfrac{1}{3}, ?0x \mapsto \tfrac{2}{9}, ?1x \mapsto \tfrac{1}{9}, ?1?0x \mapsto \tfrac{2}{27}, ?1?1x \mapsto \tfrac{1}{27}, \ldots\}$$

What Else in Paper

- Completeness for infinitary distributions works out under additional principles, such as cancellativity or impersonalization
- Unlike completeness, definability works for arbitrary signatures
- Definability in game-theoretic terms via victorious strategies

Open Problem

- Our completeness proof works for any finitary signatures
- It crucially relies on the fact that terms can be normalized by pushing infinite sums upwards using

$$\left(\sum_{n\in I} p_n \cdot M_n\right) * \left(\sum_{m\in J} q_m \cdot N_m\right) \rightsquigarrow \sum_{n\in I, m\in J} (p_n \cdot q_m) \cdot N_n * M_m$$

- With countable signatures, this is not possible, e.g.

$$\mathsf{Age?}\left(x_0, \tfrac{1}{2}x_0 + \tfrac{1}{2}x_1, \tfrac{1}{3}x_0 + \tfrac{1}{3}x_1 + \tfrac{1}{3}x_2, \ldots\right)$$

  there is no normal form in this sense
- So, do we have completeness in this case? We do not know

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More on Tensors

- Tensors of monads/theories go back to Freyd[3]
- They were utilized in computer science for commutative combination of effects[4] and extended beyond finitary case
- Often used as a tool to enforce quotienting by trace semantics[5]
  (e.g. certain stack monad $\mathbf{T}$ is behind deterministic push-down automata (pda) – to obtain nondeterministic pda, one needs $\mathbf{T} \otimes \mathcal{P}_{<\omega}$)

[3] Freyd, "Algebra valued functors in general and tensor products in particular", 1966.

[4] Hyland, Plotkin, and Power, "Combining Computational Effects: Commutativity & Sum", 2002.

[5] Goncharov, Milius, and Silva, "Towards a Uniform Theory of Effectful State Machines", 2020.